KUWAIT UNIVERSITY

Dept. of Math. & Comp. Sci. April 12, 2009

Math-102 Duration : 90 minutes

First Midterm

Calculators and mobile phones are NOT allowed.
Answer all of the following questions.

1. (3 pts) Use logarithmic differentiation to find $\dfrac{dy}{dx}$ if

$$y = \frac{\sin^{-1}(x^2)\sqrt{3^x+1}}{(x+1)^{-x}\cosh x}$$

2. (3 pts) Find the exact value of

$$\cos\left(2\tan^{-1}\left(\frac{1}{3}\right) - \frac{\pi}{2}\right)$$

3. (3 pts) Show that

$$\cosh(\ln(\tan x)) = \csc(2x), \qquad \text{for } 0 < x < \frac{\pi}{2}$$

4. (4+3+3 pts) Evaluate the following integrals

(i) $$\int \left(\frac{2e^x}{4+e^{2x}} + \frac{e^{2x}}{4+e^{2x}}\right) dx$$

(ii) $$\int (\coth x)\,\ln(\sinh x)\,dx$$

(iii) $$\int \frac{dx}{\sqrt{9^x - 4}}$$

5. (3 pts) Let g be the inverse function of f and $h(x) = xg^2(x)$. If $f(1) = 3$ and $f'(1) = 2$ then find $h'(3)$.

6. (3 pts) Solve for x,

$$\log_{\frac{1}{e}}(4 - 3x) < 1$$

1. $\ln y = \ln(\sin^{-1}(x^2)) + \dfrac{1}{2}\ln(3^x+1) + x\ln(x+1) - \ln\cosh x \;\Rightarrow$

$$y' = \left[\frac{1}{\sin^{-1}(x^2)}\frac{2x}{\sqrt{1-x^4}} + \frac{1}{2}\cdot\frac{3^x\ln 3}{3^x+1} + \ln(x+1) + \frac{x}{x+1} - \frac{\sinh x}{\cosh x}\right]\cdot y$$

2. $u = \tan^{-1}\left(\dfrac{1}{3}\right) \;\Leftrightarrow\; \tan u = \dfrac{1}{3} \;\Rightarrow\; \cos u = \dfrac{3}{\sqrt{10}},\ \sin u = \dfrac{1}{\sqrt{10}} \Rightarrow$

$$\cos\left(2\tan^{-1}\left(\frac{1}{3}\right) - \frac{\pi}{2}\right) = \cos\left(2u - \frac{\pi}{2}\right) = \sin(2u) = 2\sin u\cos u = \frac{3}{5}$$

3. $\cosh(\ln(\tan x)) = \dfrac{\tan x + \cot x}{2} = \dfrac{\sin^2 x + \cos^2 x}{2\sin x\cos x} = \dfrac{1}{\sin 2x} = \csc 2x$

4. **(i)** If $u = e^x$ and $v = e^{2x}$ then

$$\int\left(\frac{2e^x}{4+e^{2x}} + \frac{e^{2x}}{4+e^{2x}}\right)dx = \int\frac{2}{4+u^2}\,du + \frac{1}{2}\int\frac{dv}{4+v}$$

$$= \tan^{-1}\frac{e^x}{2} + \frac{1}{2}\ln\left(4+e^{2x}\right) + C$$

(ii) If $u = \ln(\sinh x)$, then $du = \coth x\,dx$ and

$$\int(\coth x)\,\ln(\sinh x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\ln^2(\sinh x) + C$$

(iii) If $u = 3^x$ then $du = 3^x\ln 3\,dx$ and

$$\int\frac{dx}{\sqrt{9^x-4}} = \int\frac{3^x\,dx}{3^x\sqrt{(3^x)^2-4}} = \frac{1}{\ln 3}\int\frac{du}{u\sqrt{u^2-4}} = \frac{1}{2\ln 3}\sec^{-1}\left(\frac{3^x}{2}\right) + C$$

5. $f(1) = 3 \;\Leftrightarrow\; g(3) = 1;\quad g'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{2}$ and
$h'(x) = g^2(x) + 2xg(x)g'(x) \quad\Rightarrow\quad h'(3) = 1^2 + 2\times 3\times 1\times\frac{1}{2} = 4$

6. $\log_{\frac{1}{e}}(4-3x) = -\ln(4-3x) < 1 \quad\Rightarrow\quad \ln(4-3x) > -1 \quad\Rightarrow\quad 4-3x > e^{-1}$
$\Rightarrow\quad x < \dfrac{4-e^{-1}}{3}$